We have also learnt in Chapter 1 that if $f: X \rightarrow Y$ such that $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \rightarrow X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Here, the domain of g = range of f and the range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Further, g is also one-one and onto and inverse of g is f. Thus, $g^{-1} = (f^{-1})^{-1} = f$. We also have

$$(f^{-1} \text{ o} f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and $$(f \text{ o} f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

Since the domain of sine function is the set of all real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range [– 1, 1]. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is [–1, 1]. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is [– 1, 1] and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is [–1, 1] and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

Remarks

(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse

Homework

Prove the following:

1. $3\sin^{-1} x = \sin^{-1} (3x - 4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

2. $3\cos^{-1} x = \cos^{-1} (4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$

3. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

4. $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Maths Class 12th

Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

❖ *Mathematics, in general, is fundamentally the science of self-evident things.* — *FELIX KLEIN* ❖

2.1 Introduction

In Chapter 1, we have studied that the inverse of a function f, denoted by f^{-1}, exists if f is one-one and onto. There are many functions which are not one-one, onto or both and hence we can not talk of their inverses. In Class XI, we studied that trigonometric functions are not one-one and onto over their natural domains and ranges and hence their inverses do not exist. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behaviour through graphical representations. Besides, some elementary properties will also be discussed.

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(476-550 A. D.)

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering.

2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \to [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \to [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{ x : x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{ x : x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$